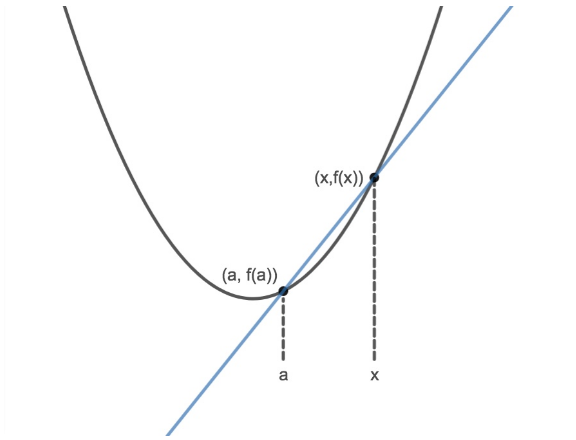
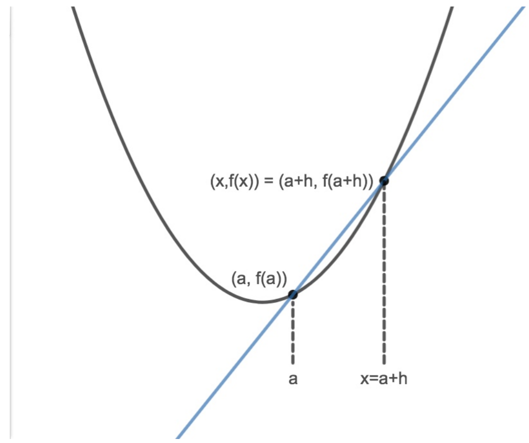
SECTION 3.1: INTRODUCTION TO DERIVATIVES

RECALL: If f is a function defined on an interval $[a,x]$, the **average rate of change of** f is given by:

$$\text{ARC}_{[a,x]} = \frac{\Delta[f(x)]}{\Delta x} = \frac{f(x)-f(a)}{x-a} = \frac{\text{change in outputs}}{\text{change in inputs}}$$

Geometrically, the average rate of change of f over $[a,x]$ is the slope of the line through the points $(a,f(a))$ and $(x,f(x))$. This line is called the **secant line** ('secant' meaning 'to cut through') and is sketched below on the left. Oftentimes, x is relabeled as $a+h$ (or a + 'a little bit.') This scenario is sketched below on the right.



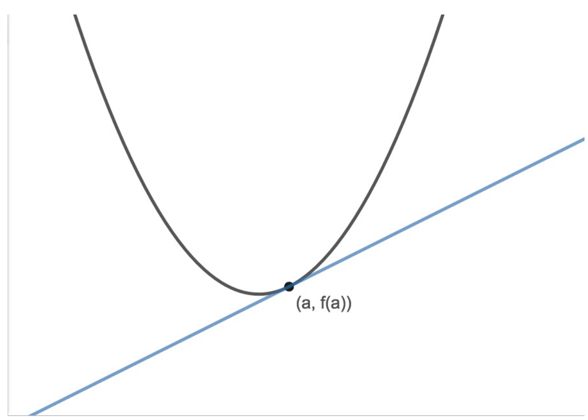


$$m_{\text{sec}} = \frac{f(x)-f(a)}{x-a}$$

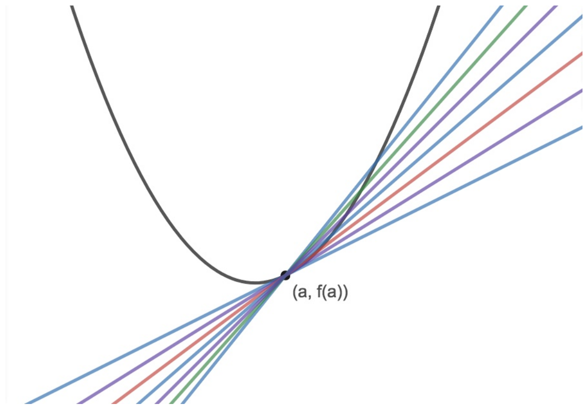
$$m_{\text{sec}} = \frac{f(a+h)-f(a)}{(a+h)-a} = \frac{f(a+h)-f(a)}{h}$$

The ARC is an example of a '**difference quotient**' and the variable 'h' is called an **increment**.

In Calculus we aim to find the slope of the elusive **tangent line** ('tangent' meaning 'to touch') as shown below:



BIG IDEA: We can **approximate** the slope of the tangent line by the slope of the secant line by choosing points 'very close to' $x = a$, as seen below. This leads us to **define** the slope of the tangent line as the **limit** of the slopes of the secant lines as $x \to a$ (or, equivalently, $h \to 0$).



DEFINITION: The slope of the tangent line at $x = a$ is $m_{\text{tan}} = \lim\limits_{x \to a} \dfrac{f(x) - f(a)}{x - a}$, provided the limit exists.

Equivalently, $m_{\text{tan}} = \lim\limits_{h \to 0} \dfrac{f(a+h) - f(a)}{h}$, provided the limit exists.

NOTE: Direct substitution always results in a $\frac{0}{0}$ indeterminate form!

EXAMPLE 1: Consider the function $f(x) = \sqrt{x}$.

1. Find the slope of the tangent line when $a = 4$ using the formula: $m_{\text{tan}} = \lim\limits_{x \to a} \dfrac{f(x) - f(a)}{x - a}$.

   Ans: $m_{\text{tan}} = \lim\limits_{x \to a} \dfrac{f(x) - f(a)}{x - a} = \lim\limits_{x \to 4} \dfrac{f(x) - f(4)}{x - 4} = \lim\limits_{x \to 4} \dfrac{\sqrt{x} - \sqrt{4}}{x - 4} = \lim\limits_{x \to 4} \dfrac{\sqrt{x} - 2}{x - 4} = \ldots = \lim\limits_{x \to 4} \dfrac{1}{\sqrt{x} + 2} = \dfrac{1}{4}$

2. Find the slope of the tangent line when $a = 4$ using the formula: $m_{\text{tan}} = \lim\limits_{h \to 0} \dfrac{f(a+h) - f(a)}{h}$.

   Ans: $m_{\text{tan}} = \lim\limits_{h \to 0} \dfrac{f(a+h) - f(a)}{h} = \lim\limits_{h \to 0} \dfrac{f(4+h) - f(4)}{h} = \lim\limits_{h \to 0} \dfrac{\sqrt{4+h} - \sqrt{4}}{h} = \lim\limits_{h \to 0} \dfrac{\sqrt{4+h} - 2}{h} = \ldots = \lim\limits_{h \to 0} \dfrac{1}{\sqrt{4+h} + 2} = \dfrac{1}{4}$

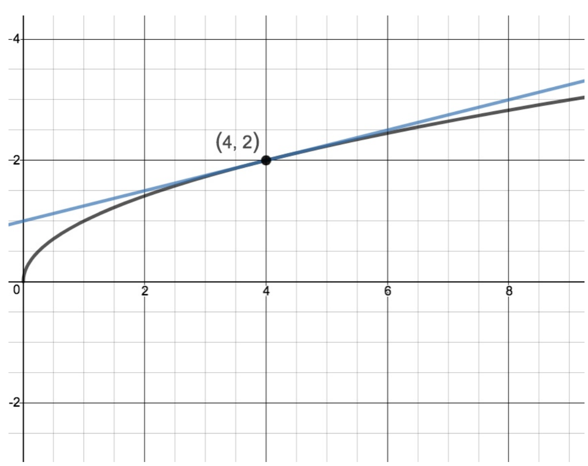
3. Write the equation of the tangent line when $a = 4$ and check your answer using a graphing utility.

   To write the equation of a line, we need a point and a slope.

   The point on the graph of $f(x) = \sqrt{x}$ corresponding to $a = 4$ is $(4, f(4)) = (4, \sqrt{4}) = (4, 2)$.

   Ans: The equation of the line through $(4, 2)$ with slope $m = \frac{1}{4}$ is $y = \frac{1}{4}(x - 4) + 2$ or $y = \frac{1}{4}x + 1$.

   Note how the tangent line becomes indistinguishable from the graph of $f(x) = \sqrt{x}$ as we near $(4, 2)$.



EXAMPLE 2: (VIDEO) For each function below:

- Find the slope of the tangent line at the indicated value using: $m_{\tan} = \lim\limits_{x \to a} \dfrac{f(x) - f(a)}{x - a}$
- Find the slope of the tangent line at the indicated value using: $m_{\tan} = \lim\limits_{h \to 0} \dfrac{f(a+h) - f(a)}{h}$
- Write the equation of the tangent line at the given value. Check your answer graphically.

1. $f(x) = 2x - x^2$ at $a = 3$.

$$\text{Ans: } m_{\tan} = \lim_{x \to 3} \frac{f(x) - f(3)}{x - 3} = \lim_{x \to 3} \frac{2x - x^2 - (-3)}{x - 3} = \ldots = -4$$

$$\text{Ans: } m_{\tan} = \lim_{h \to 0} \frac{f(3+h) - f(3)}{h} = \lim_{h \to 0} \frac{2(3+h) - (3+h)^2 - (-3)}{h} = \ldots = -4$$

Tangent line: $y = -4(x - 3) + (-3)$ which simplifies to $y = -4x + 9$.

2. $f(x) = \dfrac{1}{x}$ at $a = \dfrac{1}{2}$.

$$\text{Ans: } m_{\tan} = \lim_{x \to \frac{1}{2}} \frac{f(x) - f(\frac{1}{2})}{x - \frac{1}{2}} = \lim_{x \to \frac{1}{2}} \frac{\frac{1}{x} - 2}{x - \frac{1}{2}} = \ldots = -4$$

$$\text{Ans: } m_{\tan} = \lim_{h \to 0} \frac{f\left(\frac{1}{2} + h\right) - f\left(\frac{1}{2}\right)}{h} = \lim_{h \to 0} \frac{\dfrac{1}{\frac{1}{2} + h} - 2}{h} = \ldots = -4$$

Tangent line: $y = -4\left(x - \frac{1}{2}\right) + (2)$ which simplifies to $y = -4x + 4$.

EXAMPLE 3: (VIDEO) For each function below:

- Find the slope of the tangent line at the indicated value using: $m_{\tan} = \lim\limits_{t \to a} \dfrac{f(t) - f(a)}{t - a}$
- Write the equation of the tangent line at the given value. Check your answer graphically.

1. $f(t) = \sin(t)$ at $a = 0$.

$$\text{Ans: } m_{\tan} = \lim_{t \to 0} \frac{f(t) - f(0)}{t - 0} = \lim_{t \to 0} \frac{\sin(t) - \sin(0)}{t} = \ldots = 1$$

Tangent line: $y = (1)(t - 0) + 0$ which simplifies to $y = t$.

2. $f(t) = \cos(t)$ at $a = 0$.

$$\text{Ans: } m_{\tan} = \lim_{t \to 0} \frac{f(t) - f(0)}{t - 0} = \lim_{t \to 0} \frac{\cos(t) - \cos(0)}{t} = \ldots = 0$$

Tangent line: $y = (0)(t - 0) + 1$ which simplifies to $y = 1$.

EXAMPLE 4: (VIDEO) For the functions below, we can't yet find the slope of the tangent line analytically.

- **Approximate** the slope of the tangent line at the indicated value by **approximating**: $m_{\tan} = \lim\limits_{x\to a} \dfrac{f(x) - f(a)}{x - a}$
- Write the equation of the tangent line at the given value. Check your answer graphically.

1. $f(x) = \ln(x)$ at $a = 1$.

   Ans: $m_{\tan} = \lim\limits_{x\to 1} \dfrac{f(x) - f(1)}{x - 1} = \ldots = \lim\limits_{x\to 1} \dfrac{\ln(x)}{x - 1} \approx 1$

   Tangent line: $y = 1(x - 1) + 0$ which simplifies to $y = x - 1$.

2. $f(x) = e^x$ at $a = 0$.

   Ans: $m_{\tan} = \lim\limits_{x\to 0} \dfrac{f(x) - f(0)}{x - 0} = \ldots = \lim\limits_{x\to 0} \dfrac{e^x - 1}{x} \approx 1$

   Tangent line: $y = 1(x - 0) + 1$ which simplifies to $y = x + 1$.

DEFINITION: The **derivative** of f at $x = a$, denoted $f'(a)$ is the slope of the tangent line at $x = a$.
We have two formulas for $f'(a)$:

$$f'(a) = \lim_{x\to a} \frac{f(x) - f(a)}{x - a} \qquad \text{or} \qquad f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

If $f'(a)$ exists, f is said to be **differentiable** at $x = a$.

With this notation, we can now write a nice formula for the tangent line:

EQUATION OF TANGENT LINE: If $f'(a)$ exists, the tangent line at $x = a$ is $y = f'(a)(x - a) + f(a)$.

We have defined $f'(a)$ to be the slope of the tangent line at a. We circle back to the concept of rate of change.

DEFINITION: If f is differentiable at a, the **instantaneous rate of change** at $x = a$, denoted $\text{IRoC}(a)$ is:

$$\text{IRoC}(a) = \lim_{x\to a} \text{ARoC}_{[a,x]} = \lim_{x\to a} \frac{f(x) - f(a)}{x - a} = f'(a)$$

The units on $f'(a)$ are $\dfrac{\text{units of outputs, } f(x)}{\text{units of inputs , } x}$

NOTE: It is helpful to keep in mind the difference between $f(a)$ and $f'(a)$:

- $f(a)$:
  - gives y-**value** on the graph of $y = f(x)$ at $x = a$: $(a, f(a))$.
  - gives the value of the **output** of a function f at $x = a$
- $f'(a)$:
  - gives the **slope of the tangent line** of the graph $y = f(x)$ at $(a, f(a))$.
  - gives the **instantaneous rate of change** of a function f at $x = a$

EXAMPLE 5: Suppose $T(t)$ represents the temperature (in °F) t hours after 6 AM on a particular day.

1. Interpret the statement: $T(2) = 20$ in terms of time and temperature.

   Ans: This means that at 8 AM (2 hours after 6 AM) the temperature is 20°F.

2. Interpret the following statements in terms of time and temperature:

   (a) $T'(2) = 4$

   Ans: This means at 8 AM, the temperature is **rising** at a rate of 4° per hour.

3. Write functional expressions equivalent to the following statements.

   (a) At 2 PM, the temperature is 36°F .

   Ans: Since 2 PM is 8 hours after 6 AM, we have $T(8) = 36$.

   (b) At 2 PM, the temperature is **falling** at a rate of 3° per hour.

   Ans: $T'(8) = -3$.

4. Suppose $T(14) = 12$ and $T'(t) = 0$ for $14 \le t \le 20$. What does this mean in terms of time and temperature?

   Ans: At 8 PM (14 hours after 6 AM), the temperature is 12°F. Since $T'(t) = 0$ for $14 \le t \le 20$, the temperature remains 12°F until 2 AM the following morning (20 hours after 6 AM).

DEFINITION:

- A **position function**, denoted $s(t)$, represents **where** an object is at time t.
- The **instantaneous velocity** of the object is $v(t) = s'(t)$.
  - The **speed** of the object is given by $|v(t)|$. This is how fast the object is traveling.
  - The sign $(+)$ or $(-)$ of $v(t)$ indicates in which **direction** the object is moving.

EXAMPLE 6: The formula $s(t) = -5t^2 + 100t$ for $0 \le t \le 20$ gives the height, $s(t)$, measured in feet, of a model rocket above the Moon's surface as a function of the time after lift-off, t, in seconds. Find and interpret $s'(15)$.

Ans:

$$s'(15) = \lim_{t\to 15} \frac{s(t) - s(15)}{t - 15} = \lim_{t\to 15} \frac{-5t^2 + 100t - 375}{t - 15} = \ldots = -50$$

15 seconds after launch, the rocket is traveling 50 feet per second **downwards**.

HOMEWORK: Section 3.1: 9 - 53 every other odd.